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QUASI-GEOSTROPHIC MOTIONS IN A ROTATING LAYER OF AN ELECTRICALLY CONDUCTING FLUID

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Large-scale nonlinear oscillations of an electrically conducting ideal fluid of varying depth are considered with the magnetic, Archimedean, and Coriolis forces taken into account. The main equations are derived from an analysis of the scales of quasi-geostrophic motions. Under the assumptions that the Rossby numbers (a measure of the ratio of the local and advective accelerations to the Coriolis acceleration) are of the same order, the problem is reduced to a system of three nonlinear equations for hydromagnetic pressure and two functions describing the magnetic field. For an infinitely long horizontal layer of an electrically conducting rotating fluid, the exact solution of the corresponding nonlinear equations and the dispersion relation are obtained under the assumption of an approximately constant slope of the upper boundary surface of the layer at a distance of the order of the wavelength.

Key words: *electrically conducting rotating fluid, quasi-geostrophic motion, Rossby numbers, nonlinear equations with partial derivatives, long waves.*

Introduction. Electromagnetic processes are known to underlie a number of phenomena. For example, electromagnetic processes in the Earth's interior lead to the occurrence of the general magnetic field of the planet. According to the conventional concepts (see, for example, [1–3]), the Earth's magnetic field is generated by motions in the fluid part of the Earth's core, but details of this process have not been studied.

There has been an increasing interest in studies of the Earth's core. This is due to the fact that the core has a significant influence on the various global geophysical phenomena and the processes that have occurred and are occurring in the Earth and which can also be manifested on its surface.

Since the electrically conducting cores of the planets, in particular, the Earth's core, have been inadequately studied, the development of planetary dynamo theory can lead to the emergence of new theories in the interpretation of magnetic measurement results, in particular, data on secular geomagnetic variations.

Electromagnetic processes in the Earth's core are related to various processes in its mantle; therefore, as noted in [4], investigation of the geomagnetic field is an essential part of the geophysical studies of the internal structure and development of the Earth.

Investigation of the structure of the Earth's interior is an important and complex problem which is solved using primarily geophysical (seismic and gravitational) methods.

The mathematical problem describing magnetic field generation by motion of an electrically conducting fluid is called the hydromagnetic dynamo problem. The idea of the hydromagnetic dynamo was first expressed by J. Larmour (1919) in explaining the origin of magnetic fields on the Sun [5]. The hydromagnetic dynamo has been investigated theoretically in studies of magnetic fields in astrophysics and geophysics [6], but this phenomenon is known to have a more general meaning in magnetohydrodynamics. Fundamental works on dynamo theory [1–3, 7] proved the existence of stationary and nonstationary solutions of the magnetic induction equation with the specification of some special type of velocity field. Thereby, the fundamental possibility of the geodynamo is proved.

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Because of the complexity of the equations describing magnetohydrodynamic processes in the Earth's core, Most of the research efforts have been directed at obtaining solutions of the Maxwell equations for specified velocity distributions. Models in which the fluid velocity is considered to be specified and only the magnetic field is determined are called kinematic models of the Earth's dynamo.

In [8], it is noted that laboratory experiments do not reproduce the generation of planetary magnetic fields by motion of an electrically conducting fluid which is sustained by the forces that are natural for these objects. The laboratory dynamo of Lowes and Wilkinson [9, 10] is an attempt to implement existing theoretical models such as the Herzenberg dynamo [11]. However, these experiments are unsuitable for verifying dynamo models in geophysics. Because of the very large dimensions of space conductors and the long duration of the phenomena, small mechanical forces of magnetic origin can make significant changes in the motion of the conductors. Therefore, before extending the results of laboratory experiments to space objects, they need to be verified. Quite often, observations of space objects replace laboratory experiments. However, interpretation of the results of space observations without preliminary mathematical analysis is insufficiently reliable. Thus, along with the need for developing physical, geophysical, and numerical methods, there is a need for performing analytical studies using the mathematical apparatus.

The systems of differential equations in partial derivatives which describe the physical phenomena considered in the present paper are nonlinear and nonautonomic and have a large dimension. Analytical, in particular, exact solutions can be constructed only for particular cases of the systems. Generally, the initial system is approximated by a simpler system which adequately describes the properties of solutions of the initial system.

In the present paper, an attempt is made to construct analytical, in particular, exact, solutions of the problem of the quasi-geostrophic motion of an electrically conducting ideal rotating fluid modeling wave motions in the Earth's fluid core bounded by the surfaces of the Earth's mantle and solid core.

According to [12], the relationship between the electromagnetic and hydrodynamic phenomena is enhanced as the linear scale of the phenomenon increases. For large-scale phenomena, this relationship can be very strong, for example, in the interior of stars and the Earth's fluid core.

Large-scale motions of an electrically conducting fluid were studied in [8, 13–16], where a model constructed in an approximation of fast rotation was considered. In this model, the force of inertia is ignored in the equation of motion and, as a result, the inertial, Alfvén, and Rossby waves are not considered. In addition, in the fast rotation limit, the velocity $\boldsymbol{v}$ is not determined uniquely but up to a certain term representing the geostrophic velocity. This is due to the fact that the geostrophic velocity does not satisfy the magnetogeostrophic equation. To overcome the above-mentioned difficulties, viscous forces are introduced and viscosity (where admissible) is ignored.

A similar model for the layer enclosed between the planes $z = 0$ and $z = d$ is analyzed in [8, 13]. In the present paper, it is assumed that boundaries of the layer are surfaces which vary in space and time; in addition, inertial forces are taken into account in the equation of motion.

1. Governing Equations. We use rectangular Cartesian coordinates $Oxyz$. By the bulk force is meant a vector $\boldsymbol{g}$ which is perpendicular to the surface $z = 0$ and points in the opposite direction to the vertical axis. The axis of rotation of the fluid coincides with the z axis, i.e., $\boldsymbol{\omega} = \boldsymbol{k}\omega$, where $\boldsymbol{\omega}$ is the angular velocity of the Earth's rotation.

We consider a rotating layer of an electrically conducting ideal incompressible fluid which is bounded from above by the solid impenetrable surface $z = -Z(x, y)$ of the Earth's mantle and from below by the surface $z = -h_b(x, y, t)$ of the Earth's solid core. In the Eulerian variables, the motion of an inviscid electrically conducting incompressible fluid rotating at angular velocity $\boldsymbol{\omega}$ is described by the following system of equations [12, 17–20]

$$\operatorname{div} \boldsymbol{v} = 0; \tag{1}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\nabla p}{\rho} - 2\,\boldsymbol{\omega} \times \boldsymbol{v} - g\boldsymbol{z} + \frac{1}{\mu\rho}\operatorname{rot} \boldsymbol{B} \times \boldsymbol{B}; \tag{2}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{rot}(\boldsymbol{v} \times \boldsymbol{B}) + \frac{1}{\sigma\mu}\,\Delta \boldsymbol{B}; \tag{3}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{4}$$

where $\boldsymbol{B}$ is the magnetic induction vector, $\boldsymbol{v}$ is the fluid velocity in the coordinate system rotating at angular velocity $\boldsymbol{\omega}$, p is the pressure, ρ is the density, and g is the acceleration due to gravity. It is assumed that the magnetic permeability μ and the electrical conductivity σ are constant.

An important property of Eq. (3) is that, as $1/(\sigma\mu) \to 0$ (infinitely conducting fluid), the field flux $\boldsymbol{B}$ through any material surface in the fluid is conserved. This implies that the field $\boldsymbol{B}$ varies as if the magnetic lines of this field are frozen in the moving material. Since in bodies of cosmic dimensions, the field lines move very slowly, it can be assumed that they are almost frozen in the material [12].

Thus, we assume that the fluid possesses high electrical conductivity such that the magnetic Reynolds number is large:

$$\mathrm{Re}_m = LU/\lambda \gg 1$$

[L and U are the characteristic size and velocity and $\lambda = 1/(\sigma\mu)$ is the magnetic diffusion coefficient]. The case $\mathrm{Re}_m \gg 1$ occurs, for example, in the Earth's fluid core. For $\mathrm{Re}_m \gg 1$, Eq. (3) becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \mathrm{rot}\,(\boldsymbol{v} \times \boldsymbol{B}). \tag{5}$$

For the problem considered, the equations of magnetohydrodynamics (1), (2), (4), and (5) in the projections onto the coordinate axes are written as

$$\begin{gathered}
\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} \\
= -\frac{1}{\rho}\frac{\partial}{\partial x}\Big(p + \frac{b^2}{2\mu}\Big) + 2\omega v_y + \frac{1}{\mu\rho}\Big(b_x\frac{\partial b_x}{\partial x} + b_y\frac{\partial b_x}{\partial y} + b_z\frac{\partial b_x}{\partial z}\Big), \\
\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} \\
= -\frac{1}{\rho}\frac{\partial}{\partial y}\Big(p + \frac{b^2}{2\mu}\Big) - 2\omega v_x + \frac{1}{\mu\rho}\Big(b_x\frac{\partial b_y}{\partial x} + b_y\frac{\partial b_y}{\partial y} + b_z\frac{\partial b_y}{\partial z}\Big), \\
\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} \\
= -\frac{1}{\rho}\frac{\partial}{\partial z}\Big(p + \frac{b^2}{2\mu}\Big) - g + \frac{1}{\mu\rho}\Big(b_x\frac{\partial b_z}{\partial x} + b_y\frac{\partial b_z}{\partial y} + b_z\frac{\partial b_z}{\partial z}\Big),
\end{gathered} \tag{6}$$

$$\begin{gathered}
\frac{\partial b_x}{\partial t} + v_x\frac{\partial b_x}{\partial x} + v_y\frac{\partial b_x}{\partial y} + v_z\frac{\partial b_x}{\partial z} - b_x\frac{\partial v_x}{\partial x} - b_y\frac{\partial v_x}{\partial y} - b_z\frac{\partial v_x}{\partial z} = 0, \\
\frac{\partial b_y}{\partial t} + v_x\frac{\partial b_y}{\partial x} + v_y\frac{\partial b_y}{\partial y} + v_z\frac{\partial b_y}{\partial z} - b_x\frac{\partial v_y}{\partial x} - b_y\frac{\partial v_y}{\partial y} - b_z\frac{\partial v_y}{\partial z} = 0; \\
\frac{\partial b_z}{\partial t} + v_x\frac{\partial b_z}{\partial x} + v_y\frac{\partial b_z}{\partial y} + v_z\frac{\partial b_z}{\partial z} - b_x\frac{\partial v_z}{\partial x} - b_y\frac{\partial v_z}{\partial y} - b_z\frac{\partial v_z}{\partial z} = 0;
\end{gathered} \tag{7}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0; \tag{8}$$

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} + \frac{\partial b_z}{\partial z} = 0, \qquad b^2 = b_x^2 + b_y^2 + b_z^2. \tag{9}$$

Let us pass from system (6)–(9) to the corresponding system in dimensionless variables. We first introduce the characteristic scales for the variation of the variables of the system. Let D be the characteristic vertical scale equal to the characteristic value of the average depth of the fluid layer $-Z(x,y)+h_b(x,y,t)$, and let L be the characteristic scale of displacement in the horizontal direction of motion. We assume that

$$\delta = D/L \ll 1.$$

We also introduce the following characteristic scales: the characteristic scale U for of the horizontal velocity component, W for the vertical velocity component, B for the quantities b_x and b_y, H for the quantity b_z, T for time t, and P for the pressure field.

In Eq. (8), the first and second terms have order $O(U/L)$; therefore, the order of the third term $O(W/D)$ is not more than $O(U/L)$. Hence, using Eq. (8), we obtain

$$W \leqslant O(\delta U).$$

Similarly, using Eq. (9), we have

$$H \leqslant O(\delta B).$$

Taking into account the relations between the scales, in system (6), (7) we transform to dimensionless variables. As a result, we have the system

$$\begin{gathered}
\frac{U}{T}\frac{\partial v_x}{\partial t} + \frac{U^2}{L}\Big(v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\Big) \\
= -\frac{1}{\rho L}\Big(P + \frac{(1+\delta^2)B^2}{2\mu}\Big)\frac{\partial}{\partial x}\Big(p + \frac{b^2}{2\mu}\Big) + 2\omega U v_y + \frac{B^2}{L\mu\rho}\Big(b_x\frac{\partial b_x}{\partial x} + b_y\frac{\partial b_x}{\partial y} + b_z\frac{\partial b_x}{\partial z}\Big), \\
\frac{U}{T}\frac{\partial v_y}{\partial t} + \frac{U^2}{L}\Big(v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\Big) \qquad (10) \\
= -\frac{1}{\rho L}\Big(P + \frac{(1+\delta^2)B^2}{2\mu}\Big)\frac{\partial}{\partial y}\Big(p + \frac{b^2}{2\mu}\Big) - 2\omega U v_x + \frac{B^2}{L\mu\rho}\Big(b_x\frac{\partial b_y}{\partial x} + b_y\frac{\partial b_y}{\partial y} + b_z\frac{\partial b_y}{\partial z}\Big);
\end{gathered}$$

$$\begin{gathered}
\frac{\delta U}{T}\frac{\partial v_z}{\partial t} + \frac{\delta U^2}{L}\Big(v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\Big) \\
= -\frac{1}{\rho D}\Big(P + \frac{(1+\delta^2)B^2}{2\mu}\Big)\frac{\partial}{\partial z}\Big(p + \frac{b^2}{2\mu}\Big) - g + \frac{\delta B^2}{L\mu\rho}\Big(b_x\frac{\partial b_z}{\partial x} + b_y\frac{\partial b_z}{\partial y} + b_z\frac{\partial b_z}{\partial z}\Big); \qquad (11) \\
\frac{B}{T}\frac{\partial b_x}{\partial t} + \frac{UB}{L}\Big(v_x\frac{\partial b_x}{\partial x} + v_y\frac{\partial b_x}{\partial y} + v_z\frac{\partial b_x}{\partial z} - b_x\frac{\partial v_x}{\partial x} - b_y\frac{\partial v_x}{\partial y} - b_z\frac{\partial v_x}{\partial z}\Big) = 0, \\
\frac{B}{T}\frac{\partial b_y}{\partial t} + \frac{UB}{L}\Big(v_x\frac{\partial b_y}{\partial x} + v_y\frac{\partial b_y}{\partial y} + v_z\frac{\partial b_y}{\partial z} - b_x\frac{\partial v_y}{\partial x} - b_y\frac{\partial v_y}{\partial y} - b_z\frac{\partial v_y}{\partial z}\Big) = 0, \\
\frac{\delta B}{T}\frac{\partial b_z}{\partial t} + \frac{\delta UB}{L}\Big(v_x\frac{\partial b_z}{\partial x} + v_y\frac{\partial b_z}{\partial y} + v_z\frac{\partial b_z}{\partial z} - b_x\frac{\partial v_z}{\partial x} - b_y\frac{\partial v_z}{\partial y} - b_z\frac{\partial v_z}{\partial z}\Big) = 0.
\end{gathered}$$

Here and below, for the dimensionless variables we use the same notation as the for dimensional ones.

From Eqs. (10), it follows that the scale of the dynamic pressure P and the magnetic pressure B^2/μ is equal to the largest value of the parameters $\rho UL/T$, ρU^2, and $2\omega\rho UL$; otherwise, the acceleration of the motion will be zero.

We simplify the examined system of differential equations. Retaining the main terms in Eq. (11), we obtain

$$\frac{\partial}{\partial z}\Big(p + \frac{b^2}{2\mu}\Big) = -\rho g$$

or, integrating with respect to z,

$$p + b^2/(2\mu) = -\rho g z + C(x, y, t).$$

Using the boundary conditions $p(x, y, -h_b) = p_0$ and $b(x, y, -h_b) = b_0$, where p_0 and b_0 are constants, we have

$$p + b^2/(2\mu) = p_0 + b_0^2/(2\mu) - \rho g(h_b + z). \qquad (12)$$

Expression (12) implies that the horizontal gradient of the hydromagnetic pressure does not depend on z:

$$\frac{\partial}{\partial x}\Big(p+\frac{b^2}{2\mu}\Big)=-\rho g\,\frac{\partial h_b}{\partial x},\qquad \frac{\partial}{\partial y}\Big(p+\frac{b^2}{2\mu}\Big)=-\rho g\,\frac{\partial h_b}{\partial y};$$

therefore, the horizontal components of the velocity and magnetic field also do not depend on z if they did not depend on z at the initial time.

Equations (10) become

$$\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}=2\omega v_y+g\,\frac{\partial h_b}{\partial x}+\frac{1}{\mu\rho}\Big(b_x\frac{\partial b_x}{\partial x}+b_y\frac{\partial b_x}{\partial y}\Big),$$

$$\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}=2\omega v_x+g\,\frac{\partial h_b}{\partial y}+\frac{1}{\mu\rho}\Big(b_x\frac{\partial b_y}{\partial x}+b_y\frac{\partial b_y}{\partial y}\Big).$$

Because v_x and v_y do not depend on z, Eq. (8) can be integrated with respect to z:

$$v_z(x,y,z,t)=-z\Big(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\Big)+a(x,y,t).$$

Using the condition of no normal velocity components on the solid surface $z=-Z$, we have

$$v_z(x,y,-Z,t)=-v_x\frac{\partial Z}{\partial x}-v_y\frac{\partial Z}{\partial y}.$$

Consequently,

$$a(x,y,t)=-v_x\frac{\partial Z}{\partial x}-v_y\frac{\partial Z}{\partial y}-Z\Big(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\Big),$$

and hence

$$v_z(x,y,z,t)=-(Z+z)\Big(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\Big)-v_x\frac{\partial Z}{\partial x}-v_y\frac{\partial Z}{\partial y}. \tag{13}$$

Using equality (13), from the boundary condition

$$v_z=-\frac{\partial h_b}{\partial t}-v_x\frac{\partial h_b}{\partial x}-v_y\frac{\partial h_b}{\partial y},\qquad z=-h_b(x,y,t)$$

we obtain

$$\frac{\partial h_b}{\partial t}+\frac{\partial}{\partial x}\,[(-Z+h_b)v_x]+\frac{\partial}{\partial y}\,[(-Z+h_b)v_y]=0.$$

We integrate Eq. (9) with respect to z:

$$b_z(x,y,z,t)=-z\Big(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\Big)+\tilde{a}(x,y,t).$$

On the surface $z=-h_b$, the following condition is satisfied:

$$b_z(x,y,-h_b,t)=b_{z0}(x,y,t).$$

Consequently,

$$\tilde{a}(x,y,t)=b_{z0}(x,y,t)-h_b\Big(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\Big),$$

and hence

$$b_z(x,y,z,t)=-(h_b+z)\Big(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\Big)+b_{z0}(x,y,t). \tag{14}$$

In view of the condition on the surface $z=-Z$

$$b_z(x,y,-Z,t)=b^{(e)}_{z0}(x,y,t),$$

equality (14) becomes

$$(h_b-Z)\Big(\frac{\partial b_x}{\partial x}+\frac{\partial b_y}{\partial y}\Big)+b^{(e)}_{z0}(x,y,t)-b_{z0}(x,y,t)=0.$$

Thus, since $\delta \ll 1$, we have the system of equations

$$\begin{aligned} &\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 2\omega v_y + g\,\frac{\partial h_b}{\partial x} + \frac{1}{\mu\rho}\Big(b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_x}{\partial y}\Big), \\ &\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -2\omega v_x + g\,\frac{\partial h_b}{\partial y} + \frac{1}{\mu\rho}\Big(b_y \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_y}{\partial y}\Big); \end{aligned} \tag{15}$$

$$\frac{\partial h_b}{\partial t} + \frac{\partial}{\partial x}\,[(-Z + h_b)v_x] + \frac{\partial}{\partial y}\,[(-Z + h_b)v_y] = 0; \tag{16}$$

$$\begin{aligned} &(-Z + h_b)\Big(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\Big) + b_{z0}^{(e)}(x, y, t) - b_{z0}(x, y, t) = 0, \\ &\frac{\partial b_x}{\partial t} + v_x \frac{\partial b_x}{\partial x} + v_y \frac{\partial b_x}{\partial y} - b_x \frac{\partial v_x}{\partial x} - b_y \frac{\partial b_x}{\partial y} = 0, \\ &\frac{\partial b_y}{\partial t} + v_x \frac{\partial b_y}{\partial x} + v_y \frac{\partial b_y}{\partial y} - b_x \frac{\partial v_y}{\partial x} - b_y \frac{\partial v_y}{\partial y} = 0. \end{aligned} \tag{17}$$

The result is a decrease in the number of dynamic equations, the required functions (due to the elimination of v_z and b_z from the equations of the initial system), and independent variables [since z is not explicitly included in (15)–(17)]. The remaining variables (v_x, v_y, b_x, b_y, and h_b) are functions of only x, y, and t.

From expressions (13) and (14), it follows that the z-components of the velocity v_z and the magnetic field b_z are functions linear in z.

The boundary conditions for Eqs. (15)–(17) are the conditions of nonpenetration through the vertical surfaces on the boundary of the region considered (if they are available) with the specified magnetic field on them:

$$v_x \cos(\boldsymbol{n}, \boldsymbol{x}) + v_y \cos(\boldsymbol{n}, \boldsymbol{y}) = 0,$$

$$b_x = b_x^{(L)}, \qquad b_y = b_y^{(L)}, \qquad (x, y) \in L$$

($\boldsymbol{n}$ is the normal to the horizontal section of the boundary of the region).

We note that, ignoring the Lorentz magnetic force, Eqs. (15) and (16) are the long-wave equations, or, according to [21], the shallow-water equations. In the presence of the magnetic field, the number of the basic shallow-water equations changes (the Maxwell equation of the magnetic field induction and the equation of the solenoidality of the magnetic field are added) and the form of the equations of motion (15), which is due to the presence of the last terms on the right sides of these equations.

2. Derivation and Solution of the Equations of Quasi-Geostrophic Motion. We introduce a function of the total depth $H = -Z + h_b$. Let the thickness of the fluid layer at rest be equal to $H_0(x, y)$. The function $H(x, y, t)$ is represented as

$$H(x, y, t) = H_0(x, y) + \eta(x, y, t) = D - Z + \eta(x, y, t), \qquad \eta \ll H_0. \tag{18}$$

Within the framework of the nonlinear equations of long-wave approximation, in view of representation (18), the examined problem is written as

$$\begin{aligned} &\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 2\omega v_y + g\,\frac{\partial \eta}{\partial x} + \frac{1}{\mu\rho}\Big(b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_x}{\partial y}\Big), \\ &\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -2\omega v_x + g\,\frac{\partial \eta}{\partial y} + \frac{1}{\mu\rho}\Big(b_x \frac{\partial b_y}{\partial x} + b_y \frac{\partial b_y}{\partial y}\Big), \\ &\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\,[(H_0 + \eta)v_x] + \frac{\partial}{\partial y}\,[(H_0 + \eta)v_y] = 0, \\ &(H_0 + \eta)\Big(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\Big) + b_{z_0}^{(e)}(x, y, t) - b_{z_0}(x, y, t) = 0, \end{aligned} \tag{19}$$

$$\frac{\partial b_x}{\partial t} + v_x \frac{\partial b_x}{\partial x} + v_y \frac{\partial b_x}{\partial y} - b_x \frac{\partial v_x}{\partial x} - b_y \frac{\partial v_x}{\partial y} = 0,$$

$$\frac{\partial b_t}{\partial t} + v_x \frac{\partial b_y}{\partial x} + v_y \frac{\partial b_y}{\partial y} - b_x \frac{\partial v_y}{\partial x} - b_y \frac{\partial v_y}{\partial y} = 0.$$

Let us make a more detailed analysis of the orders of the terms in system (19). Let L be the linear scale, U the velocity scale, B the magnetic-field scale, T the time scale, and N the scale of the ordinate of the surface $\eta(x, y, t)$. We introduce the following dimensionless variables:

$$x', \quad y', \quad t', \quad v'_x, \quad v'_y, \quad b'_x, \quad b'_y, \quad \eta', \quad x = Lx', \quad y = Ly',$$

$$t = Lt', \quad v_x = Uv'_x, \quad v_y = Uv'_y, \quad b_x = Bv'_x, \quad b_y = Bb'_y, \quad \eta = N\eta'.$$

Then, the basic equations (19) are written as follows (primes are omitted):

$$\begin{aligned} &\varepsilon_T \frac{\partial v_x}{\partial t} + \varepsilon\Big(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y}\Big) - v_y = \frac{gN}{L\alpha U} \frac{\partial \eta}{\partial x} + \frac{1}{\mu\rho}\Big(\frac{B}{U}\Big)^2 \varepsilon\Big(b_x \frac{\partial b_x}{\partial x} + b_y \frac{\partial b_x}{\partial y}\Big), \\ &\varepsilon_T \frac{\partial v_y}{\partial t} + \varepsilon\Big(v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\Big) + v_x = \frac{gN}{L\alpha U} \frac{\partial \eta}{\partial y} + \frac{1}{\mu\rho}\Big(\frac{B}{U}\Big)^2 \varepsilon\Big(b_x \frac{\partial b_y}{\partial x} + b_y \frac{\partial b_y}{\partial y}\Big); \end{aligned} \tag{20}$$

$$\begin{aligned} &\varepsilon_T F \frac{\partial \eta}{\partial t} + \varepsilon F\Big(v_x \frac{\partial \eta}{\partial x} + v_y \frac{\partial \eta}{\partial y}\Big) - v_x \frac{\partial}{\partial x}\Big(\frac{Z}{D}\Big) - v_y \frac{\partial}{\partial y}\Big(\frac{Z}{D}\Big) \\ &\quad + \Big(1 + \varepsilon F\eta - \frac{Z}{D}\Big)\Big(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\Big) = 0; \end{aligned} \tag{21}$$

$$\begin{aligned} &\Big(1 + \varepsilon F\eta - \frac{Z}{D}\Big)\Big(\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y}\Big) + b^{(e)}_{z_0}(x, y, t) - b_{z_0}(x, y, t) = 0, \\ &\varepsilon_T \frac{\partial b_x}{\partial t} + \varepsilon\Big(v_x \frac{\partial b_x}{\partial x} + v_y \frac{\partial b_x}{\partial y} - b_x \frac{\partial v_x}{\partial x} - b_y \frac{\partial v_x}{\partial y}\Big) = 0, \\ &\varepsilon_T \frac{\partial b_y}{\partial t} + \varepsilon\Big(v_x \frac{\partial b_y}{\partial x} + v_y \frac{\partial b_y}{\partial y} - b_x \frac{\partial v_y}{\partial x} - b_y \frac{\partial v_y}{\partial y}\Big) = 0. \end{aligned} \tag{22}$$

Here

$$\varepsilon_T = \frac{1}{T\alpha}, \quad \varepsilon = \frac{U}{L\alpha}, \quad F = \frac{f^2 L^2}{gD}, \quad \alpha = 2\omega.$$

Next, we set

$$\frac{gN}{L\alpha U} = 1. \tag{23}$$

Then, $N = L\alpha U/g$. From the above analysis, it follows that $B^2/(\mu\rho U^2) = O(1)$ and $F = O(1)$. In addition, we assume that

$$Z/D = \varepsilon\eta_b, \qquad b^{(e)}_{z_0} - b_{z_0} = \varepsilon b_h \tag{24}$$

(η_b and b_h have the order of unity). Condition (24) means that the Rossby number ε, though small, is still so large that the motion differs significantly from strictly geostrophic motions.

The Rossby numbers ε_T and ε are a measure of the ratio of the local and advective accelerations to the Coriolis acceleration. The ratio of the local acceleration to the advective one is determined by the parameter

$$\frac{\varepsilon_T}{\varepsilon} = \frac{L}{UT}.$$

If this parameter is large, the equations are linear, in essence, i.e., the local time derivative dominates over the nonlinear advective terms. We assume that the nonlinear terms are as important as the local acceleration. In other words, we assume that $\varepsilon_T/\varepsilon = 1$, i.e., we consider the cases where the time of advection L/U has the same order of magnitude as the time scale of local changes.

We using conditions (23) and (24) and applying the rot operator to Eqs. (20). As a result, in view of Eq. (21) for $\varepsilon_T = \varepsilon$, from system (20)–(22) as a first approximation ($\varepsilon = 0$), we obtain the equations

$$\Big(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y}\Big)(\Omega - F\eta + \eta_b) = M\Big(b_x \frac{\partial \zeta}{\partial x} + b_y \frac{\partial \zeta}{\partial y}\Big); \tag{25}$$

$$v_x = \frac{\partial \eta}{\partial y}, \qquad v_y = -\frac{\partial \eta}{\partial x}, \qquad \Omega = -\Delta\eta, \qquad \zeta = \frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y}; \tag{26}$$

$$\frac{\partial b_x}{\partial x} + \frac{\partial b_y}{\partial y} = 0; \tag{27}$$

$$\begin{aligned} &\frac{\partial b_x}{\partial t} + v_x \frac{\partial b_x}{\partial x} + v_y \frac{\partial b_x}{\partial y} - b_x \frac{\partial v_x}{\partial x} - b_y \frac{\partial v_x}{\partial y} = 0, \\ &\frac{\partial b_y}{\partial t} + v_x \frac{\partial b_y}{\partial x} + v_y \frac{\partial b_y}{\partial y} - b_x \frac{\partial v_y}{\partial x} - b_y \frac{\partial v_y}{\partial y} = 0, \end{aligned} \tag{28}$$

where $M = B^2/(\mu\rho U^2)$. In the absence of the magnetic field, this approximation in hydrodynamics is called quasi-geostrophic [21].

Solutions of the system of nonlinear equations (25)–(28) are sought in the form of a dependence of the magnetic field component on the function η:

$$b_x = f_1(\eta), \qquad b_y = f_2(\eta). \tag{29}$$

Then, Eqs. (25) and (28) in terms of the function η become

$$\begin{aligned} &\Big(\frac{\partial}{\partial t} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \eta}{\partial x}\frac{\partial}{\partial y}\Big)(-\Delta\eta - F\eta + \eta_b) \\ &= M\Big[f_2''\Big(f_1\Big(\frac{\partial \eta}{\partial x}\Big)^2 + f_2 \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\Big) - f_1''\Big(f_1 \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + f_2\Big(\frac{\partial \eta}{\partial y}\Big)^2\Big) \\ &\quad + f_2'\Big(f_1 \frac{\partial^2 \eta}{\partial x^2} + f_2 \frac{\partial^2 \eta}{\partial x\,\partial y}\Big) - f_1'\Big(f_1 \frac{\partial^2 \eta}{\partial x\,\partial y} + f_2 \frac{\partial^2 \eta}{\partial y^2}\Big)\Big]; \end{aligned} \tag{30}$$

$$\begin{aligned} &f_1' \frac{\partial \eta}{\partial t} - f_1 \frac{\partial^2 \eta}{\partial x\,\partial y} - f_2 \frac{\partial^2 \eta}{\partial y^2} = 0, \\ &f_2' \frac{\partial \eta}{\partial t} + f_1 \frac{\partial^2 \eta}{\partial x^2} + f_2 \frac{\partial^2 \eta}{\partial x\,\partial y} = 0. \end{aligned} \tag{31}$$

The terms in the expression $-\Delta\eta - F\eta + \eta_b$ are completely determined by the relative motion.

Thus, the problem of determining the quasi-geostrophic motion is reduced to the solution of three nonlinear equations for the perturbation of the surface η (or, what is the same, for hydromagnetic pressure) and for the functions $f_1(\eta)$ and $f_2(\eta)$ describing the magnetic field. Having obtained the solutions η, f_1, and f_2 of Eqs. (30) and (31), we can determine the velocity components v_x and v_y and the field components b_x and b_y from relations (26) and (29).

The solution η is sought in the form

$$\eta = A\,\mathrm{e}^{i(kx+ly-\sigma t)}\,.$$

Then, system (30), (31) becomes

$$\begin{aligned} i\Big(-\sigma(k^2+l^2) + \sigma F + l\frac{\partial \eta_b}{\partial x} - k\frac{\partial \eta_b}{\partial y}\Big) &= \frac{1}{\mu\rho}\Big(f_1''(klf_1 + l^2 f_2) - f_2''(k^2 f_1 - klf_2)\Big)A\,\mathrm{e}^{i(kx+ly+\sigma t)} \\ &\quad + \Big(f_1'(klf_1 + l^2 f_2) - f_2'(k^2 f_1 + klf_2)\Big); \end{aligned} \tag{32}$$

$$-i\sigma f_1' + klf_1 + l^2 f_2 = 0, \qquad i\sigma f_2' + k^2 f_1 + klf_2 = 0. \tag{33}$$

The functions $f_1(\eta)$ and $f_2(\eta)$ in analytical form are found from system (33). Eliminating the function f_2 from this system, we obtain

$$f_2 = \frac{i\sigma}{l^2}\, f_1' - \frac{k}{l}\, f_1.$$

Then, for the function $f_1(\eta)$ we have the equation

$$f_1''(\eta) = 0$$

and its general solution

$$f_1(\eta) = C_1\eta + C_2. \tag{34}$$

Consequently,

$$f_2(\eta) = \frac{i\sigma C_1}{l^2} - \frac{kC_2}{l} - \frac{kC_1}{l}\,\eta. \tag{35}$$

In view of expressions (34) and (35), Eq. (32) becomes

$$-\sigma(k^2+l^2) + \sigma F + l\,\frac{\partial\eta_b}{\partial x} - k\,\frac{\partial\eta_b}{\partial y} = \frac{\sigma C_1^2 M}{l^2}\,(k^2+l^2). \tag{36}$$

We note that Eq. (27), which is the condition of the quasi-solenoidality of the magnetic field, is satisfied identically.

Thus, the following conclusion is valid. For an infinitely long horizontal layer of an electrically conducting rotating fluid for $\nabla\eta_b = \text{const}$, which is equivalent to the assumption of an approximately constant slope of the surface $z = -Z(x,y)$ at a distance of the order of the wavelength, we have the exact solution of the system of nonlinear equations (25)–(28)

$$\eta = A\,\mathrm{e}^{i(kx+ly-\sigma t)}, \qquad v_x = \frac{\partial\eta}{\partial y}, \qquad v_y = -\frac{\partial\eta}{\partial x},$$

$$b_x = C_1\eta + C_2, \qquad b_y = \frac{i\sigma C_1}{l^2} - \frac{kC_2}{l} - \frac{kC_1}{l}\,\eta.$$

Equation (36) implies the dispersion relation

$$\sigma = \frac{k\,\partial\eta_b/\partial y - l\,\partial\eta_b/\partial x}{F - (1+k^2/l^2)(l^2 + C_1^2 M)}.$$

We note that, in the case $C_1 M = 0$, the dispersion relation has the same form as for a low-frequency Rossby wave in an electrically nonconducting fluid. In both cases, waves with higher frequencies are not taken into account because of the *a priori* assumption of the quasi-geostrophic nature of the motion.

The main characteristics of the motion can be written as follows:

$$\eta(x,y,t) = A\,\mathrm{e}^{\sigma_2 t}\cos(kx+ly-\sigma_1 t),$$

$$b_x = (-a\sin(kx+ly-\sigma_1 t) + b\cos(kx+ly-\sigma_1 t))A\,\mathrm{e}^{\sigma_2 t} + C_2,$$

$$b_y = -\frac{k}{l}\,(-a\sin(kx+ly-\sigma_1 t) + b\cos(kx+ly-\sigma_1 t))A\,\mathrm{e}^{\sigma_2 t} - \Big(\frac{\sigma_1 a + \sigma_2 b}{l^2} - \frac{kC_2}{l}\Big),$$

$$v_x = -lA\,\mathrm{e}^{\sigma_2 t}\sin(kx+ly-\sigma_1 t), \qquad v_y = kA\,\mathrm{e}^{\sigma_2 t}\sin(kx+ly-\sigma_1 t).$$

Here

$$\sigma_1 = \frac{(k\,\partial\eta_b/\partial y - l\,\partial\eta_b/\partial x)(FM - (1+k^2/l^2)(l^2M + a^2 - b^2))M}{(FM - (1+k^2/l^2)(l^2M + a^2 - b^2))^2 + 4a^2b^2(1+k^2/l^2)^2M^4},$$

$$\sigma_2 = \frac{2ab(1+k^2/l^2)(k\,\partial\eta_b/\partial y - l\,\partial\eta_b/\partial x)M^3}{(FM - (1+k^2/l^2)(l^2M + a^2 - b^2))^2 + 4a^2b^2(1+k^2/l^2)^2}.$$

In this case, $a, b, \sigma_1, \sigma_2 \in \mathbb{R}$, $C_1 = a + ib$, and $\sigma = \sigma_1 + i\sigma_2$. The sign of σ_2 depends on the sign of the expression $ab(k\,\partial\eta_b/\partial y - l\,\partial\eta_b/\partial x)$. For a bounded solution to exist, it is necessary that the values of a and b satisfy the inequality

$$ab\Big(k\,\frac{\partial\eta_b}{\partial y}-l\,\frac{\partial\eta_b}{\partial x}\Big)<0.$$

Projecting the magnetic induction equations onto the vertical axis, it is possible to establish the relationship between the amplitude of the external magnetic field, the mantle relief, and the oscillation amplitude of the boundary of the Earth's solid core. Indeed, in the problem considered, Eq. (7) becomes

$$\frac{\partial b_{z0}^{(e)}}{\partial t}+\frac{\mathcal{D}(\eta,b_{z0}^{(e)})}{\mathcal{D}(x,y)}+\frac{i\sigma C_1}{l^2}\,\frac{\mathcal{D}(\partial\eta/\partial y,Z)}{\mathcal{D}(x,y)}=0.$$

For a field periodic along the horizontal coordinates and time $b_{z0}^{(e)}$, i.e., for example, for $b_{z0}^{(e)}=B\,\mathrm{e}^{i(kx+ly-\sigma t)}$, the above equation implies that

$$A=\frac{B}{C_1((k/l)\,\partial Z/\partial y-\partial Z/\partial x)}.$$

Thus, the analytical solutions presented here allow one to determine the effect of the mantle relief and the dynamics of the Earth's solid core on the magnetohydrodynamic characteristics of the wave process in the fluid core. The results of these studies can be used in astrophysics and geophysics, in particular, in studies of the processes occurring in the Earth's liquid core and the interior of stars.

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